

# NONEXISTENCE OF CUSP CROSS-SECTION OF ONE-CUSPED COMPLETE COMPLEX HYPERBOLIC MANIFOLDS II

YOSHINOBU KAMISHIMA

**ABSTRACT.** Long and Reid have shown that some compact flat 3-manifold cannot be diffeomorphic to a cusp cross-section of any complete finite volume 1-cusped hyperbolic 4-manifold. Similar to the flat case, we give a negative answer that there exists a 3-dimensional closed Heisenberg infranilmanifold whose diffeomorphism class cannot be arisen as a cusp cross-section of any complete finite volume 1-cusped complex hyperbolic 2-manifold. This is obtained from the formula by the characteristic numbers of bounded domains related to the Burns-Epstein invariant on strictly pseudo-convex  $CR$ -manifolds [1],[3]. This paper is a sequel of our paper[11].

## INTRODUCTION

We shall consider whether every Heisenberg infranilmanifold can be arisen, up to diffeomorphism, as a cusp cross-section of a complete finite volume 1-cusped complex hyperbolic manifold. Long and Reid considered the problem that every compact Riemannian flat manifold is diffeomorphic to a cusp cross-section of a complete finite volume 1-cusped hyperbolic manifold. They have shown it is false for some compact flat 3-manifold [15]. We shall give a negative answer similarly to the flat case.

**Theorem.** *Any 3-dimensional closed Heisenberg infranilmanifold with non-trivial holonomy cannot be diffeomorphic to a cusp cross-section of any complete finite volume 1-cusped complex hyperbolic 2-manifold.*

McReynolds informed us that W. Neumann and A. Reid have obtained the similar result.

## 2. HEISENBERG INFRANILMANIOLD

Let  $\langle z, w \rangle = \bar{z}_1 \cdot w_1 + \bar{z}_2 \cdot w_2 + \cdots + \bar{z}_n \cdot w_n$  be the Hermitian inner product defined on  $\mathbb{C}^n$ . The Heisenberg nilpotent Lie group  $\mathcal{N}$  is the product  $\mathbb{R} \times \mathbb{C}^n$  with group law:

$$(2.1) \quad (a, z) \cdot (b, w) = (a + b - \operatorname{Im}\langle z, w \rangle, z + w).$$

---

*Date:* February 2, 2008.

*1991 Mathematics Subject Classification.* 53C55, 57S25, 51M10.

*Key words and phrases.* Real, Complex Hyperbolic manifold, Flat manifold, Heisenberg infranilmanifold, Cusp, Group extension, Seifert fibration.

It is easy to see that  $\mathcal{N}$  is 2-step nilpotent, i.e.  $[\mathcal{N}, \mathcal{N}] = (\mathbb{R}, 0) = \mathbb{R}$ , which is the central subgroup  $\mathcal{C}(\mathcal{N})$  of  $\mathcal{N}$ . This induces a central group extension:  $1 \rightarrow \mathcal{C}(\mathcal{N}) \rightarrow \mathcal{N} \xrightarrow{P} \mathbb{C}^n \rightarrow 1$ . Let  $\text{Iso}(\mathbb{H}_{\mathbb{C}}^{n+1})$  be the full group of the isometries of the complex hyperbolic space  $\mathbb{H}_{\mathbb{C}}^{n+1}$ . It is isomorphic to  $\text{PU}(n+1, 1) \rtimes \langle \tau \rangle$  where  $\tau$  is the (anti-holomorphic) involution induced by the complex conjugation. The Heisenberg rigid motions is defined as a subgroup of the stabilizer  $\text{Iso}(\mathbb{H}_{\mathbb{C}}^{n+1})_{\infty}$  at the point at infinity  $\infty$ .

**Definition 2.1.** *The group of Heisenberg rigid motions  $E^{\tau}(\mathcal{N})$  is defined to be  $\mathcal{N} \rtimes (\text{U}(n) \rtimes \langle \tau \rangle)$ . A Heisenberg infranilmanifold (respectively orbifold) is a compact manifold (respectively orbifold)  $\mathcal{N}/\pi$  such that  $\pi$  is a torsionfree (not necessarily torsionfree) discrete cocompact subgroup of  $E^{\tau}(\mathcal{N})$ .*

### 3. CR-STRUCTURE ON $S^{2n+1} - S^{2n-1}$

The sphere complement  $S^{2n+1} - S^{2n-1}$  is a spherical CR manifold with the transitive group  $\text{Aut}_{CR}(S^{2n+1} - S^{2n-1})$  of CR transformations which is isomorphic to the unitary Lorentz group  $\text{U}(n, 1)$ . Note that  $S^{2n+1} - S^{2n-1}$  is identified with the  $(2n+1)$ -dimensional Lorentz standard space form  $V_{-1}^{2n+1}$  of constant sectional curvature  $-1$ . The center  $\mathcal{Z}\text{U}(n, 1)$  of  $\text{U}(n, 1)$  is  $S^1$ . Then  $V_{-1}^{2n+1}$  is the total space of the principal  $S^1$ -bundle over the complex hyperbolic space:  $S^1 \rightarrow (\text{U}(n, 1), V_{-1}^{2n+1}) \xrightarrow{\nu} (\text{PU}(n, 1), \mathbb{H}_{\mathbb{C}}^n)$ . If  $\omega_{\mathbb{H}}$  is the connection form of the above principal bundle, then it is a contact form on  $V_{-1}^{2n+1}$  such that  $\text{Null } \omega_{\mathbb{H}}$  is a CR structure. Note that  $d\omega_{\mathbb{H}} = \nu^* \Omega_{\mathbb{H}}$  up to constant factor for the Kähler form  $\Omega_{\mathbb{H}}$  on  $\mathbb{H}_{\mathbb{C}}^n$ . Since  $\text{U}(n, 1) = S^1 \cdot \text{SU}(n, 1)$ , the above equivariant principal bundle induces the following commutative fibrations:

$$(3.1) \quad \begin{array}{ccccc} \mathbb{Z} & \longrightarrow & (\hat{\text{S}}\text{U}(n, 1), \tilde{V}_{-1}^{2n+1}) & \xrightarrow{\hat{\nu}} & (\text{PU}(n, 1), \mathbb{H}_{\mathbb{C}}^n) \\ \downarrow & & \downarrow & & \parallel \\ \mathbb{Z}/n+1 & \longrightarrow & (\text{SU}(n, 1), V_{-1}^{2n+1}) & \xrightarrow{\nu} & (\text{PU}(n, 1), \mathbb{H}_{\mathbb{C}}^n). \end{array}$$

Here  $\hat{\text{S}}\text{U}(n, 1)$  is a lift of  $\text{SU}(n, 1)$  associated to the covering  $\mathbb{Z} \rightarrow \tilde{V}_{-1}^{2n+1} \rightarrow V_{-1}^{2n+1}$ . For a discrete subgroup  $G \subset \text{PU}(n+1, 1)$  such that  $\mathbb{H}_{\mathbb{C}}^{n+1}/G$  is a complete finite volume complex hyperbolic orbifold, let  $\hat{G} \subset \hat{\text{S}}\text{U}(n, 1)$  be a lift where  $1 \rightarrow \mathbb{Z} \rightarrow \hat{G} \rightarrow G \rightarrow 1$  is an exact sequence. Then  $S^1 \rightarrow \tilde{V}_{-1}^{2n+1}/\hat{G} \xrightarrow{\hat{\nu}} \mathbb{H}_{\mathbb{C}}^{n+1}/G$  is an injective Seifert fibration (i.e. the singular fiber bundle with typical fiber is  $S^1$ . The exceptional fiber is also a circle.)

### 4. BURNS AND EPSTEIN'S FORMULA

In general, the Heisenberg infranilmanifold or its two fold cover at least admits a spherical CR-structure, see Definition 2.1. In [2], Burns and Epstein obtained the CR-invariant  $\mu(M)$  on the 3-dimensional strictly pseudoconvex CR-manifolds  $M$  provided that the holomorphic line bundle is trivial.

Let  $X$  be a compact strictly pseudoconvex complex 2-dimensional manifold with smooth boundary  $M$ . Then they have shown the following equality in [3]:

$$(4.1) \quad \int_X c_2 - \frac{1}{3}c_1^2 = \chi(X) - \frac{1}{3} \int_X \bar{c}_1^2 + \mu(M).$$

Here  $\bar{c}_1$  is a lift of  $c_1$  by the inclusion  $j^* : H^2(X, M; \mathbb{R}) \rightarrow H^2(X; \mathbb{R})$ .

## 5. GEOMETRIC BOUNDARY

**5.1. One-cusped complex hyperbolic 2-manifold.** Let  $E^\tau(\mathcal{N})$  be the group of Heisenberg rigid motions on the 3-dimensional Heisenberg nilpotent Lie group  $\mathcal{N}$  and  $L : E^\tau(\mathcal{N}) \rightarrow U(1) \rtimes \langle \tau \rangle$  the holonomy homomorphism. Suppose that  $M = \mathcal{N}/\Gamma$  is realized as a cusp cross-section of a complete finite volume one-cusped complex hyperbolic 2-manifold  $W = \mathbb{H}_{\mathbb{C}}^2/G$ . Put  $\bar{W} = \mathbb{H}_{\mathbb{C}}^2/G - M \times (0, \infty)$  so that  $\partial \bar{W} = M$ . Then  $\bar{W}$  is homotopic to  $W$  and  $M$  is viewed as a boundary of  $\text{Int} \bar{W}$  which supports a complete complex hyperbolic structure. The holonomy group  $L(\Gamma)$  of a 3-dimensional compact Heisenberg non-homogeneous infranilmanifold  $M = \mathcal{N}/\Gamma$  is a cyclic subgroup of order 2, 3, 4, 6 of  $U(1)$  or  $L(\Gamma)$  is isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset U(1) \rtimes \langle \tau \rangle$ , see [4], [16] for the classification. If  $M$  has the holonomy  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , then  $G$  has nontrivial summand in  $\langle \tau \rangle$  of  $\text{Iso}(\mathbb{H}_{\mathbb{C}}^2) = \text{PU}(2, 1) \rtimes \langle \tau \rangle$ . The two fold cover  $W/G \cap \text{PU}(2, 1)$  is still a one-cusped complex hyperbolic manifold for which the cusp cross-section is the two fold cover of  $M$  whose holonomy group becomes  $\mathbb{Z}/2 \subset U(1)$ . When the holonomy group belongs to  $U(1)$ , the spherical  $CR$ -structure on  $M$  is canonically induced from the complex hyperbolic structure on  $W$ . (Note that  $\tau$  does not preserve the  $CR$ -structure bundle.)

**5.2. Integral of  $\bar{c}_1^2$ .** Let  $p : \tilde{W} \rightarrow W$  be the finite covering, say of order  $\ell$ , whose induced covering  $\tilde{M}$  of  $M$  is now a (homogeneous) nilmanifold (using the separability argument if necessary). Possibly it consists of a finite number of such nilmanifolds. Since  $W$  admits a complete Einstein-Kähler metric, we know that  $c_2 - \frac{1}{3}c_1^2 = 0$ . Moreover, since  $\tilde{M}$  is a spherical  $CR$  manifold with trivial holomorphic line bundle, it follows that  $\mu(\tilde{M}) = 0$ . As in §4, let  $j^* : H^2(\bar{W}, M; \mathbb{R}) \rightarrow H^2(\bar{W}; \mathbb{R}) = H^2(W; \mathbb{R})$  be the map such that  $j^*\bar{c}_1(\bar{W}) = c_1(W)$ . Applying (4.1) to  $\tilde{W}$ , we have  $\chi(\tilde{W}) = \frac{1}{3} \int_{\tilde{W}} \bar{c}_1^2$ . As  $p^*(\bar{c}_1(\bar{W})) = \bar{c}_1(\tilde{W})$  by naturality and  $p_*[\tilde{W}] = \ell[\bar{W}]$ , it follows that  $\int_{\tilde{W}} \bar{c}_1^2 = \langle \bar{c}_1^2(\tilde{W}), [\tilde{W}] \rangle = \langle \bar{c}_1^2(\bar{W}), \ell[\bar{W}] \rangle$ . Since  $\chi(\tilde{W}) = \ell\chi(W)$ ,  $3\chi(W) = \langle \bar{c}_1^2(\bar{W}), [\bar{W}] \rangle$ .

**Proposition 5.1.** *If  $M = \mathcal{N}/\Gamma$  is realized as a cusp cross-section of a complete finite volume one-cusped complex hyperbolic 2-manifold  $W = \mathbb{H}_{\mathbb{C}}^2/G$ , then  $\bar{c}_1^2(\bar{W})$  is an integer in  $H^4(\bar{W}, M; \mathbb{Z}) = \mathbb{Z}$ .*

**5.3. Torsion element in  $M$ .** Given a  $CR$ -structure on  $M$ , there is the canonical splitting  $TM \otimes \mathbb{C} = B^{1,0} \oplus B^{0,1}$  where  $B^{1,0}$  is the holomorphic line bundle. Since  $M$  is an infranilmanifold but not homogeneous,  $B^{1,0}$  is nontrivial, i.e.  $c_1(B^{1,0}) \neq 0$ . (In fact, it is a torsion element in  $H^2(M : \mathbb{Z})$ , because the  $\ell$ -fold covering  $\bar{M}$  has the trivial holomorphic bundle.) The spherical  $CR$  manifold  $M$  has a characteristic  $CR$  vector field  $\xi$ . If  $\epsilon^1$  is the vector field on  $M$  pointing outward to  $W$ , then the distribution  $\langle \epsilon^1, \xi \rangle$  generates a trivial holomorphic line bundle  $T\mathbb{C}^{1,0}$  on  $M$  for which  $TW \otimes \mathbb{C}|_M = B^{1,0} + T\mathbb{C}^{1,0} \oplus B^{0,1} + T\mathbb{C}^{0,1}$ . As  $i^*(c_1(W)) = c_1(B^{1,0} + T\mathbb{C}^{1,0}) = c_1(B^{1,0})$  and  $\ell \cdot c_1(B^{1,0}) = 0$ , we have  $j^*\beta = \ell \cdot c_1(W)$  for some integral class  $\beta \in H^2(\bar{W}, M : \mathbb{Z})$ .

**5.4.  $H_1(M : \mathbb{Z})$ .** Let  $1 \rightarrow \Delta \rightarrow \Gamma \rightarrow F \rightarrow 1$  be the group extension of the fundamental group  $\Gamma = \pi_1(M)$  where  $\Delta$  is the maximal normal nilpotent subgroup and  $F \cong \mathbb{Z}_\ell$  ( $\ell = 2, 3, 4, 6$ ) or  $F \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . Recall that  $\Delta$  is generated by  $\{a, b, c\}$  where  $[a, b] = aba^{-1}b^{-1} = c^k$  for some  $k > 0$ . It follows that  $\Delta/[\Delta, \Delta] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k$ . Let  $\gamma$  be an element of  $\Gamma$  which maps to a generator of  $\mathbb{Z}_\ell$ . A calculation shows that mod  $[\Delta, \Delta]$ ,

$$(5.1) \quad \begin{aligned} \gamma a \gamma^{-1} &= a^{-1}, & \gamma b \gamma^{-1} &= b^{-1} & (\ell = 2), \\ \gamma a \gamma^{-1} &= b, & \gamma b \gamma^{-1} &= a^{-1}b^{-1} & (\ell = 3), \\ \gamma a \gamma^{-1} &= b, & \gamma b \gamma^{-1} &= a^{-1} & (\ell = 4), \\ \gamma a \gamma^{-1} &= b, & \gamma b \gamma^{-1} &= a^{-1}b & (\ell = 6). \end{aligned}$$

When  $F = \mathbb{Z}_2 \times \mathbb{Z}_2$ , let  $\delta$  be an element of  $\Gamma$  which goes to another generator of  $F$ . Then  $\gamma a \gamma^{-1} = a$ ,  $\gamma b \gamma^{-1} = b^{-1}$  mod  $[\Delta, \Delta]$ . In view of the above relation (5.1),  $\gamma$  (also  $\delta$ ) becomes a torsion element of order  $m$  in  $\Gamma/[\Gamma, \Gamma]$  where  $m$  is divisible by  $\ell$ . As  $\Gamma$  is generated by  $\{a, b, c, \gamma\}$  or  $\{a, b, c, \gamma, \delta\}$ , it follows that

$$(5.2) \quad H_1(M : \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_m \oplus \left\{ \begin{array}{ll} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & (\ell = 2) \\ \mathbb{Z}_3 & (\ell = 3) \\ \mathbb{Z}_2 & (\ell = 4) \\ 1 & (\ell = 6) \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \end{array} \right\}.$$

In any case, if  $\mathcal{N}/\Gamma$  has a nontrivial holonomy group  $F$ , then  $H_1(M : \mathbb{Z})$  is a torsion group.

**5.5. Intersection number.** Put  $\bar{H}^2(\bar{W}, M : \mathbb{Z}) = H^2(\bar{W}, M : \mathbb{Z})/\text{Tor}$  where Tor is the torsion subgroup. We have a nondegenerate inner product  $\bar{H}^2(\bar{W}, M : \mathbb{Z}) \times \bar{H}^2(\bar{W}, M : \mathbb{Z}) \rightarrow \mathbb{Z}$  defined by the intersection form

$$(x, y) = \langle x \cup y, [\bar{W}] \rangle.$$

Denote by  $\bar{W} \# \pm \mathbb{CP}^2$  the connected sum of  $\bar{W}$  with  $\mathbb{CP}^2 \# -\mathbb{CP}^2$  if necessary. We can assume that  $(, )$  is an indefinite form of odd type, i.e. there are nonzero elements  $x, y \in \bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z}) = \bar{H}^2(\bar{W}, M) + \langle 1 \rangle + \langle -1 \rangle$

such that  $(x, x)$  is odd and  $(y, y) = 0$ . By  $\langle \pm 1 \rangle$  we shall mean that it is generated by either  $x_+$  or  $x_-$  of  $\bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$  such that  $(x_\pm, x_\pm) = \pm 1$  respectively. Moreover by the classification of nondegenerate indefinite inner product cf. [10], there is an isomorphism preserving the inner product from  $\bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$  onto

$$(5.3) \quad m\langle 1 \rangle \oplus n\langle -1 \rangle = \langle 1 \rangle_1 \oplus \cdots \oplus \langle 1 \rangle_m \oplus \langle -1 \rangle_1 \oplus \cdots \oplus \langle -1 \rangle_n$$

for  $(m, n \neq 0)$ . Here  $\langle \pm 1 \rangle_i$  is the  $i$ -th copy of  $\langle \pm 1 \rangle$ . Consider the commutative diagram:

$$(5.4) \quad \begin{array}{ccccc} H^2(\bar{W}, M : \mathbb{Z}) & \xrightarrow{j^*} & H^2(\bar{W} : \mathbb{Z}) & \xrightarrow{i^*} & H^2(M : \mathbb{Z}) \\ D \downarrow & & D \downarrow & & D \downarrow \\ H_2(\bar{W} : \mathbb{Z}) & \xrightarrow{j_*} & H_2(\bar{W}, M : \mathbb{Z}) & \xrightarrow{\partial} & H_1(M : \mathbb{Z}). \end{array}$$

It follows from (5.2) that  $j_* : \bar{H}_2(\bar{W} : \mathbb{Z}) \rightarrow \bar{H}_2(\bar{W}, M : \mathbb{Z})$  is injective and is isomorphic if  $\mathbb{Z}$  replaces  $\mathbb{R}$ . Similarly note that  $j_* : \bar{H}_2(\bar{W} \# \pm \mathbb{CP}^2 : \mathbb{Z}) \rightarrow \bar{H}_2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$  is injective (and an isomorphism for the coefficient  $\mathbb{R}$ ). Identified the generators of  $\bar{H}_2(\bar{W} \# \pm \mathbb{CP}^2 : \mathbb{Z})$  with the basis (5.3) of  $\bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$ , we may choose the generators  $[V_i] \in \bar{H}_2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$  such that

$$(5.5) \quad j_*(\langle \pm 1 \rangle_i) = \ell_i[V_i]$$

for some  $\ell_i \in \mathbb{Z}$ .

**5.6. Canonical bundle.** The circle (line) bundle  $L : S^1 \rightarrow \tilde{V}_1/\hat{G} \rightarrow \mathbb{H}_{\mathbb{C}}^2/G = W$  is represented by the Kähler form  $\Omega$  of the Kähler-hyperbolic metric, i.e.  $[\Omega] = c_1(L) \in H^2(W : \mathbb{Z})$ . Hence  $W = \mathbb{H}_{\mathbb{C}}^2/G$  is projective-algebraic, i.e.  $W \subset \mathbb{CP}^N$  so  $c_1(W)$  can be represented by  $c_1([V])$  for some divisor  $V$  in  $W$ , i.e.  $D(c_1(W)) = [V] \in \bar{H}_2(\bar{W}, M : \mathbb{Z})$ , compare [7]. Embed  $V$  into  $\bar{W} \# \pm \mathbb{CP}^2$  and suppose that

$$[V] = \sum_i a_i[V_i] \in \bar{H}_2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z}).$$

As  $D \circ i^* c_1(W) = \partial[V]$ , it follows  $\ell \partial([V]) = 0$  by the argument of § 5.3. We observe that  $\partial[V]$  maps into  $\mathbb{Z}_m$  in  $H_1(M : \mathbb{Z})$  (cf. (5.2)) and so does each  $\partial[V_i]$ . It may occur that  $\partial a_i[V_i] = \partial a_j[V_j]$  for some  $i, j$ . So we can write  $[V] = k[V_1] + j_* x$  where  $x \in H_2(\bar{W} \# \pm \mathbb{CP}^2 : \mathbb{Z})$  and  $V_1$  satisfies that

- (1)  $\partial \bar{V}_1 = S^1$  and  $\ell[S^1] = 0$  in  $\mathbb{Z}_m \subset H_1(M : \mathbb{Z})$ .
- (2)  $\ell$  is minimal with respect to (1).
- (3)  $(k, \ell)$  is relatively prime.

**5.7. Realization of  $\bar{c}_1$ .** As  $\ell \partial[V_1] = 0$  in  $H_1(M : \mathbb{Z})$ , there is a surface  $U$  in  $W$  whose cycle  $[U] \in H_2(\bar{W} \# \pm \mathbb{CP}^2 : \mathbb{Z})$  represents  $j_*[U] = \ell[V_1]$ .

Let  $[U] = a_1\langle \pm 1 \rangle_1 + a_2\langle \pm 1 \rangle_2 + \cdots + a_s\langle \pm 1 \rangle_s$ . Then,  $\ell[V_1] = a_1\ell_1[V_1] + a_2\ell_2[V_2] + \cdots + a_s\ell_s[V_s]$ . Since each  $[V_i]$  is a generator of  $\bar{H}_2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$ , it follows that  $\ell = a_1\ell_1$  and  $a_j = 0$  ( $j \neq 1$ ). Hence  $[U] = a_1\langle \pm 1 \rangle_1$ . On

the other hand, note that  $\langle \pm 1 \rangle_1$  is a cycle of  $\bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z})$  for which  $j_*(\langle \pm 1 \rangle_1) = \ell_1[V_1]$  by (5.5). Noting that  $\ell$  is minimal by Property (2) of § 5.6,  $\ell_1$  is divisible by  $\ell$ . Therefore  $\ell_1 = \pm \ell$  and  $a_1 = \pm 1$  so that  $[U] = \pm \langle \pm 1 \rangle$ . In particular, the intersection number

$$(5.6) \quad [U] \cdot [U] = \pm 1.$$

Put  $y = \frac{k}{\ell}[U] + x \in H_2(\bar{W} \# \pm \mathbb{CP}^2 : \mathbb{R})$ . Calculate

$$\begin{aligned} y \cdot y &= \frac{k^2}{\ell^2}[U] \cdot [U] + \frac{2k}{\ell}[U] \cdot x + x \cdot x \\ &= \pm \frac{k^2}{\ell^2} + \frac{2k}{\ell}[U] \cdot x + x \cdot x, \end{aligned}$$

$$(5.7) \quad \ell(y \cdot y) = \pm \frac{k^2}{\ell} \pmod{\mathbb{Z}}$$

Noting that  $(k, \ell) = 1$  by Property (3) of § 5.6, if  $\ell \neq 1$ ,  $y \cdot y$  cannot be an integer.

As  $j_*(\frac{k}{\ell}[U]) = k[V_1]$ , note that  $j_*y = k[V_1] + j_*x = [V]$ . Consider the following diagram:

$$(5.8) \quad \begin{array}{ccc} \bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2 : \mathbb{R}) & \xrightarrow{j^*} & \bar{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{R}) \\ \parallel & & \parallel \\ \bar{H}_2(\bar{W} : \mathbb{R}) + \langle 1 \rangle + \langle -1 \rangle & \xrightarrow{j_* + \text{id}} & \bar{H}_2(\bar{W}, M : \mathbb{R}) + \langle 1 \rangle + \langle -1 \rangle. \end{array}$$

Let  $y = y_0 + t\langle 1 \rangle + s\langle -1 \rangle$  for some  $y_0 \in \bar{H}_2(\bar{W} : \mathbb{R})$ ,  $s, t \in \mathbb{R}$ . As  $j_*y = [V]$ , it follows that  $[V] = j_*y_0 + t\langle 1 \rangle + s\langle -1 \rangle$ . Noting  $[V] \in \bar{H}_2(\bar{W}, M : \mathbb{Z})$ , we have that  $[V] = j_*y_0$  and  $t = s = 0$ . In particular, this implies that  $y = y_0 \in \bar{H}_2(\bar{W} : \mathbb{R})$ . Using the commutative diagram (5.4) and by the fact  $D(c_1(W)) = [V]$ , the element  $D^{-1}(y) \in H^2(\bar{W}, M : \mathbb{R})$  satisfies that  $j^*(D^{-1}(y)) = c_1(W)$ .

On the other hand, recall from the argument of [3] that the integral  $\langle \bar{c}_1^2(\bar{W}), [\bar{W}] \rangle$  does not depend on the choice of lift  $\bar{c}_1(\bar{W})$  to  $c_1(W)$ , so we can choose  $\bar{c}_1(\bar{W}) = D^{-1}(y) \in H^2(\bar{W}, M : \mathbb{R})$  (cf. § 5.2). By definition,  $y \cdot y = \langle \bar{c}_1(\bar{W})^2, [\bar{W}] \rangle$  which is an integer by Proposition 5.1. This contradiction proves **Theorem**.

**Remark 5.2.** *Neumann and Reid have shown that if an infranil 3-manifold arises as a cusp cross-section of a 1-cusped complex hyperbolic 2-manifold, then the rational Euler number must be 1/3-integral. There are infranilmanifolds which do not satisfy this condition.*

## REFERENCES

- [1] O. Biquard and M. Herzlich, “A Burns-Epstein invariant for ACHE 4-manifolds,” *Duke Math. Jour.*

- [2] D. Burns and C. L. Epstein, "A global invariant for three-dimensional  $CR$ -structure," *Invent. Math.*, (1988), 333-348.
- [3] D. Burns and C. L. Epstein, "Characteristic numbers of bounded domains," *Acta Math.*, (1990), 29-71.
- [4] K. Dekimpe, 'Almost-Bieberbach Groups: Affine and Polynomial Structures,' *Springer-Verlag, Lecture Notes in Math. 1639*, 1996.
- [5] E. Falbel and J. R. Parker, "The geometry of the Eisenstein-Picard modular group," Preprint.
- [6] W. Goldman, 'Complex Hyperbolic Geometry,' Oxford Univ. Press (1999).
- [7] P. Griffiths and J. Harris, 'Principles of Algebraic Geometry' *John Wiley & Sons, Inc.* 1978.
- [8] N. Gusevskii and J. R. Parker, "Representations of free Fuchsian groups in complex hyperbolic space," *Topology* 39, (2000), 33-60.
- [9] G.C. Hamrick and D.C. Royster, "Flat Riemannian manifolds are boundaries," *Invent. Math.* vol. 66 (2), 405-413, 1982.
- [10] D. Husemoller and J. Milnor, 'Symmetric Bilinear Forms,' *Springer, Berlin* 1973.
- [11] Y. Kamishima, "Cusp cross-sections of hyperbolic orbifolds by Heisenberg nilmanifolds I," *to appear in Geom. Dedicata*.
- [12] Y. Kamishima, K.B. Lee and F. Raymond, "The Seifert construction and its application to infranilmanifolds," *Quart. J. Math. Oxford* (2) vol. 34 (1983), 433-452.
- [13] Y. Kamishima and T. Tsuboi, " $CR$ -structures on Seifert manifolds," *Invent. Math.*, vol. 104 (1991), pp. 149-163.
- [14] D.D. Long and A.W. Reid, "All flat manifolds are cusps of hyperbolic orbifolds", *Algebraic and Geometric Topology*. vol. 2 (2002) 285-296.
- [15] D.D. Long and A.W. Reid, "On the geometric boundaries of hyperbolic 4-manifolds", *Geometry and Topology*. vol. 4 (2000) 171-178.
- [16] D. B. McReynolds, "Peripheral separability and cusps of arithmetic hyperbolic orbifolds", *Algebr. and Geom. Topol.* 2004 (4), 721-755.
- [17] D. Toledo, "Representations of surface groups in complex hyperbolic space," *Jour. of Diff. Geom.* 29, (1989), 125-133.

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, MINAMI-OHSAWA  
 1-1, HACHIOJI, TOKYO 192-0397, JAPAN  
*E-mail address:* kami@comp.metro-u.ac.jp